

Causality in dense matter

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Abstract

The possibility of non-causal signal propagation is examined for various theories of dense matter. This investigation requires a discussion of definitions of causality, together with interpretations of spacetime position. Specific examples are used to illustrate the satisfaction or violation of causality in realistic calculations.

I. INTRODUCTION

An important question for any theory of dense matter is whether that theory (or an approximate calculation which uses it) permits the propagation of information faster than the speed of light [1]. The requirement of Poincaré invariance is not sufficient to guarantee causality, and in any event practical calculations involve approximations which may also lead to non-causal properties. The uncertainty principle leads to additional problems of interpretation.

The simplest criterion can be obtained by examining sound propagation. In the long-wavelength limit, small density fluctuations can cause isentropic pressure-wave propagation whose velocity is given by

$$v_s^2 = c^2 \left(\frac{dp}{d\epsilon} \right)_{|s}, \quad (1)$$

where p is the pressure and ϵ the energy per unit volume. The wave equation follows from considering energy momentum conservation $T^{\mu\nu}_{,\nu} = 0$ for a linear isentropic perturbation of the pressure in a perfect fluid. The sound speed follows from the equation of state. The presence of c is shown explicitly: p and ϵ may or may not depend explicitly upon c . Most of the discussion in the literature concentrates on determining whether specific theories satisfy $(dp/d\epsilon)_{|s} < 1$. For a system of non-interacting relativistic fermions, $p = \frac{1}{3}\epsilon$. However, in a mean-field approximation, Zel'dovich [2] showed that ϵ can approach p from below for a system of fermions interacting via repulsive vector meson exchange. Bludman and Ruderman [3–5] also studied questions of causality within the context of meson-nucleon field theory. Among other things, they found that the group velocity can exceed the speed of light in vacuum $((dp/d\epsilon)_{|s} > 1)$, but that real information propagation does not violate causality as long as the index of refraction $n(\omega)$ satisfies the Kramers-Kronig dispersion relation.

It would be useful to gain some insight as to the behavior of dense systems which employ particle degrees of freedom. Indeed, dense-matter calculations which start from a field theory

typically end up with models in which particles interact directly. In addition, it turns out that many calculations of the equation of state of nuclear or neutron matter still satisfy the condition $(dp/d\epsilon)_{|_s} < 1$, even though they are based on nonrelativistic quantum mechanics.

The primary motivation for this paper is therefore twofold:

1. Examine causal and non-causal behavior in theories involving directly interacting particles.
2. Understand why nonrelativistic calculations work as well as they do with respect to causality, in spite of their only approximate connection to the principles of special relativity.

To do this, it will be necessary to discuss various definitions of causality itself, as well as what is meant by spacetime position in a given theory. This discussion is presented first, followed by an examination of specific examples.

II. CAUSALITY AND SPACETIME POSITION

In relativistic quantum mechanics there are two distinct types of causality. The first is the requirement that there is a well posed initial value problem. We call this Cauchy causality. The second is the requirement that no signal can propagate faster than the speed of light. We call this Einstein causality.

A violation of Cauchy causality is a serious problem. In a relativistic quantum theory the Poincaré invariance implies the existence of a unitary representation of the Poincaré group, $U(\Lambda, a)$ that acts on the Hilbert space. The time evolution subgroup

$$U(t) := U(I, (t, 0, 0, 0)) \tag{2}$$

relates a state $|\psi(t)\rangle$ at time t to an initial state $|\psi(0)\rangle$ at time $t = 0$:

$$|\psi(t)\rangle = U(t)|\psi(0)\rangle. \tag{3}$$

This shows explicitly that Cauchy causality is a consequence of Poincaré invariance. The group properties ensure that this condition holds in all inertial coordinate systems:

$$U(\Lambda, a)|\psi(t)\rangle = U(\Lambda^0_0 t)U(\Lambda, a)|\psi(0)\rangle \quad (4)$$

which shows that the transformed final state is also uniquely determined by time evolving the transformed initial state. This shows that Cauchy causality in all inertial coordinates systems is a consequence of Poincaré invariance. Cauchy causality is independent of any considerations concerning Einstein causality.

This is distinct from the situation in relativistic classical physics, where Cauchy causality and Poincaré covariance imply Einstein causality. To see this, assume that events can be precisely identified. The Poincaré group is defined as the group of transformations that preserve the proper time between space-time events. The Poincaré group provides an invariant classification of events that are past, future, and spacelike, relative to a given event. The temporal order of spacelike separated events is coordinate system dependent. Assume that the theory satisfies Cauchy causality in all coordinate systems related by Poincaré transformations. By contradiction, assume that the theory also violates Einstein causality. Violating Einstein causality means that it is possible to send a signal from event A to event B where A and B are events with a relative spacelike separation. Since these events are spacelike separated it is possible to find a coordinate systems where event B occurs before event A . This leads to a violation of Cauchy causality in the new coordinate system, contradicting the assumption that the theory violates Einstein causality. Thus, we conclude that if we can precisely measure all spacetime events and the theory satisfies Cauchy causality in all coordinate systems related by Poincaré transformation, then the theory must also satisfy Einstein causality.

In a classical *dynamical* theory, the situation is not as clear as suggested by the previous paragraph. For instance, it is known that is impossible to construct an interacting canonical Hamiltonian theory with a non-trivial representation of the Poincaré group (with the Lie algebra realized in terms of classical Poisson brackets) that has spacetime events that

transform covariantly [6]. This gives some clues concerning the difficulties that occur in the quantum case.

In the quantum case, the Poincaré symmetry is preserved, but the concept of an event that occurs at a fixed spacetime point cannot be precisely formulated in terms of particle degrees of freedom. Specifically, one cannot use the complete set of commuting operators that define the state of a single particle to construct a set of commuting Hermitian operators whose eigenvalues transform like the spacetime coordinates of an event. More specifically, the uncertainty principle in relativistic quantum mechanics does not permit one to localize a particle with arbitrary precision. Its position can only be determined up to an uncertainty on the scale of the particle's Compton wavelength. A local field operator is needed for a precise formulation of Einstein causality.

All of these issues can be easily illustrated for the case of a free spinless quantum particle of mass m . For a free particle, the momentum eigenstates

$$|\mathbf{p} : \rangle \quad \langle \mathbf{p} | \mathbf{p}' \rangle = (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{p}') \quad (5)$$

form a complete set of states. The unitary representation of the Poincaré group,

$$U(\Lambda, a)|\mathbf{p}\rangle = \sqrt{\frac{\omega_m(\mathbf{p}')}{\omega_m(\mathbf{p})}} e^{-i\Lambda p \cdot a} |\mathbf{p}'\rangle, \quad (6)$$

is determined up to an overall phase by unitarity, the mass and spin of the particle, and the requirement that $p := (\omega_m(\mathbf{p}), \mathbf{p})$ transforms like a four vector.

This representation of the Poincaré group contains the time evolution subgroup corresponding to a free particle of mass m . It leads to a well posed initial-value problem that is consistent in any frame of reference. The single-particle dynamics thus satisfies Cauchy causality and Poincaré invariance.

Classically, an event (\mathbf{x}, t) transforms like a four-vector under Poincaré transformations. In order to investigate Einstein causality, assume by contradiction that it is possible to find a single particle state $|\mathbf{x}, t\rangle$ corresponding to the particle being found at the point \mathbf{x} at a given time t . The requirement that $|\mathbf{x}, t\rangle$ transform like a classical event is

$$U(\Lambda, a)|\mathbf{x}, t\rangle = |\mathbf{x}', t'\rangle, \quad (7)$$

where

$$(\mathbf{x}', t') := \Lambda(\mathbf{x}, t) + a. \quad (8)$$

It follows from (6) and (7) using spacetime translations that

$$\langle \mathbf{p} | \mathbf{x}, t \rangle = \langle \mathbf{p} | U(I, (\mathbf{x}, t)) | \mathbf{0}, 0 \rangle = e^{i\mathbf{p} \cdot \mathbf{x} - i\omega_m(\mathbf{p})t} \langle \mathbf{p} | \mathbf{0}, 0 \rangle \quad (9)$$

and using Lorentz transformation that

$$\langle \mathbf{p} | \mathbf{0}, 0 \rangle = \langle \mathbf{p} | U(B(p)) | \mathbf{0}, 0 \rangle = \sqrt{\frac{m}{\omega_m(\mathbf{p})}} \langle \mathbf{p} = \mathbf{0} | \mathbf{x} = \mathbf{0}, t = 0 \rangle = N \sqrt{\frac{m}{\omega_m(\mathbf{p})}} \quad (10)$$

where N is a normalization constant. Combining (9) and (10) implies the state $\langle \mathbf{p} | \mathbf{x} \rangle$ necessarily has the form of a plane wave state

$$\langle \mathbf{p} | \mathbf{x}, t \rangle = N \sqrt{\frac{m}{\omega_m(\mathbf{p})}} e^{i\mathbf{p} \cdot \mathbf{x} - i\omega_m(\mathbf{p})t}. \quad (11)$$

To check Einstein causality we calculate the overlap of states corresponding to different space-time points:

$$\begin{aligned} \langle \mathbf{x}, t | \mathbf{x}', t' \rangle &= |N|^2 \frac{1}{(2\pi)^3} \int \frac{d\mathbf{p}}{2\omega_m(\mathbf{p})} e^{i\mathbf{p} \cdot (\mathbf{x}' - \mathbf{x}) - \omega_m(\mathbf{p})(t' - t)} \\ &= -i|N|^2 D^-(x' - x), \end{aligned} \quad (12)$$

where $D^-(x)$ is the negative frequency part of the Pauli-Jordan commutator function, which has the form [7]

$$\begin{aligned} D^-(x) &:= \frac{i}{(2\pi)^3} \int \frac{d\mathbf{k}}{2\omega_m(\mathbf{k})} e^{-ik \cdot (x-y)} \\ &= \frac{1}{4\pi} \epsilon(x^0) \delta(\tau) - \frac{m\theta(\tau^2)}{8\pi\tau} [\epsilon(x^0) J_1(m\tau) - iN_1(m\tau)] + \frac{m\theta(-\tau^2)}{4\pi^2 \sqrt{-\tau^2}} \theta K_1(m\sqrt{-\tau^2}), \end{aligned} \quad (13)$$

and τ is the proper time between the events. This does not vanish for spacelike separation ($\tau^2 < 0$). The Bessel function $K_1(x)$ has the asymptotic form

$$K_1(x) \rightarrow \sqrt{\frac{\pi}{2x}} e^{-x} (1 + \frac{3}{8x} + \dots), \quad (14)$$

which decays exponentially for large x . This suggests an apparent violation of Einstein causality over a range given by the Compton wavelength of the particle. Note that what appears in the exponent is $-m\sqrt{-\tau^2}$, which involves the invariant spacelike separation. The actual spatial separation can be much larger than the Compton wavelength, provided the time difference between the events is also large. The problem is not with Einstein causality; it is related to the non-existence of a position operator that transforms as the space components of a four vector, or, equivalently, to the non-existence of a state vector representing a point particle.

The problem is that if one interprets the Fourier transform of the momentum space wave function as a position probability amplitude, then the position operator $\mathbf{x} = i\nabla_p$ is related to the generator of Lorentz boosts by equation (6), which also must be expressible in terms of the derivatives with respect to the momentum. For spinless particles, this relation is

$$\mathbf{x} := -\frac{1}{2}\left\{\frac{1}{H}, \mathbf{K}\right\}, \quad (15)$$

with slightly more complicated expressions for particles with spin. The transformation properties of the Hamiltonian H (zeroeth component of a four vector) and the boost generator \mathbf{K} ($0i$ components of a rank two antisymmetric tensor) determine the transformation properties of \mathbf{x} under Lorentz transformation.

To see that the problems with defining a position operator have nothing to do with Einstein causality, consider the free local scalar field $\phi(x)$. This operator satisfies the local commutation relations

$$[\phi(x), \phi(y)]_- = [\phi^\dagger(x), \phi^\dagger(y)]_- = [\phi^\dagger(x), \phi(y)]_- = 0 \quad (16)$$

when $x - y$ is spacelike, and the covariance conditions

$$U(\Lambda, a)\phi(x)U^\dagger(\Lambda, a) = \phi(\Lambda, x + a); \quad (17)$$

$$U(\Lambda, a)\phi^\dagger(x)U^\dagger(\Lambda, a) = \phi^\dagger(\Lambda, x + a). \quad (18)$$

The field operator can be used to construct observables

$$\phi(f) = \int d^4x f(x) \phi(x), \quad (19)$$

corresponding to a space-time region \mathcal{O} where $f(x) = 0$ for $x \notin \mathcal{O}$. The underlying quantum theory satisfies Einstein causality (16). However, if we define the state $|x\rangle := \phi(x)|0\rangle$ it follows from (17) that $|x\rangle$ satisfies (7). By direct computation,

$$\langle x|y\rangle = -iD^-(x-y), \quad (20)$$

where $\tau^2 = -(x-y)^2$. This expression is identical (up to a constant multiplicative factor) to (12). In this case the result is derived in local field theory, which clearly satisfies Einstein causality. The key observation is that $|x\rangle$ is a single particle state, but the argument x does not correspond to a precise space-time position of the particle.

What is relevant in the case of the field is that antiparticle degrees of freedom are needed ensure the commutation relations (16). The point is that the relevant local observables are the fields - not the particle excitation of the field. Although it is possible to localize the field, the single particle eigenstates are not localizable, even for the structureless particles of a free field theory. What distinguishes a particle theory from a field theory is that it is a theory of an infinite number of degrees of freedom. Particle theories do not contain enough operators to separate points in arbitrarily small spacetime regions.

Thus, we see that in a theory of particles or fields, the particle degrees of freedom cannot be used as a reliable test of Einstein causality when the separation between particles is on the order of a Compton wavelength of the particle. Nevertheless, it is possible to establish a violation of Einstein causality if it occurs on a scale significantly larger than a Compton wavelength of the particle. Certainly spacetime positions of particles are determined experimentally to within uncertainties, although large compared to the particles Compton wavelength, that are sufficiently small to localize the particle within experimental uncertainties.

If we start with a localized disturbance, and evolve it in time with a non-local Hamiltonian, then we can test to see the extent to which the disturbance propagates in a manner

that violates Einstein causality. Specifically, if non-causal effects exist on scales beyond the uncertainties in position, then the theory can ultimately lead to non-causal effects.

We now illustrate how the addition of an interaction affects Einstein causality. The simplest illustration is to consider the first order effect due to the interaction in a Bakamjian-Thomas [8] two-body model [9]. The effect of the interaction can be treated using perturbation theory. Let $H = H_0 + V$, where V is the difference between the full Bakamjian-Thomas Hamiltonian and the kinematic Hamiltonian. The time evolution operator can be expanded in powers of the interaction using the formula

$$e^{-iHt} = e^{-iH_0t} + \sum_{n=1}^{\infty} S_n(t), \quad (21)$$

where

$$S_0(t) = e^{-iH_0t} \quad S_k(t) = -i \int_0^t e^{-iH_0(t-s)} V S_{k-1}(s) ds. \quad (22)$$

The first order correction is

$$S_1(t) = -i \int_0^t e^{-iH_0(t-s)} V e^{-iH_0s} ds. \quad (23)$$

If we let \mathbf{x}_1 and \mathbf{x}_2 be the Fourier transforms of the single particle momenta and let

$$e^{iH_0t} = e^{i(H_1+H_2)t} = U_1(t)U_2(t), \quad (24)$$

then we can write

$$\begin{aligned} \langle \mathbf{x}_1, \mathbf{x}_2 | S_1(t) | \mathbf{x}_1', \mathbf{x}_2' \rangle &= -i \int_0^t ds \int \langle \mathbf{x}_1 | U_1(t-s) | \mathbf{x}_1'' \rangle \langle \mathbf{x}_2 | U_2(t-s) | \mathbf{x}_2'' \rangle d\mathbf{x}_1'' d\mathbf{x}_2'' \\ &\quad \times \langle \mathbf{x}_1'', \mathbf{x}_2'' | V | \mathbf{x}_1''', \mathbf{x}_2''' \rangle d\mathbf{x}_1''' d\mathbf{x}_2''' \\ &\quad \times \langle \mathbf{x}_1''' | U_1(t-s) | \mathbf{x}_1' \rangle \langle \mathbf{x}_2''' | U_2(t-s) | \mathbf{x}_2' \rangle. \end{aligned} \quad (25)$$

In this form, it is easy to examine the causality properties. The operators $U_i(t)$ are free-particle time evolution operators. The coordinates \mathbf{x}_i describe the position of the particle up to an uncertainty that is about the size of the Compton wavelength of the particle. The matrix element $\langle \mathbf{x}_i | U_i(t) | \mathbf{x}_i' \rangle$ can be computed:

$$\langle \mathbf{x} | U_1(t - t') | \mathbf{x}' \rangle = 2 \frac{\partial}{\partial t'} D^-(x' - x), \quad (26)$$

which again vanishes exponentially when $|\mathbf{x}_i - \mathbf{x}_i'|^2 - t^2$ is much larger than the square of a Compton wavelength. The interaction V in (25) is instantaneous. Thus propagation from \mathbf{x}_i' to \mathbf{x}_i''' in a time s is causal up to uncertainties in position. The propagation from \mathbf{x}_i''' to \mathbf{x}_i'' is instantaneous over the range of the interaction, and finally the propagation from \mathbf{x}_i'' to \mathbf{x}_i in time $t - s$ is causal up to uncertainties in position. It follows that the range of the interaction sets the scale on which non-causal effects occur in perturbation theory. If the range of the interaction is much larger than the Compton wavelength of one of the particles, then it should in principle be possible to detect such non-causal effects experimentally. This analysis is based on perturbation theory and only gives an indication of what can happen. Exact calculations can be easily performed.

Many of these issues arise in a many-body system. Questions of testing causality are intertwined with questions related to the interpretation of position variables. These issues are more complicated if the particles are detected in the nuclear medium.

Although it is worthwhile to understand these issues in a many-body problem, for the purpose of testing for violations of Einstein causality in particle theories, it is sensible to utilize an idealized model where all of the questions of interpretation become irrelevant. Consider a dilute gas of particles and imagine a long chain of successive scattering events where the mean separation of the particles is much larger than either the range of the interaction or the Compton wavelength of the particles. If each successive interaction leads to a small violation of causality, then it follows that this violation can be amplified to any desired amount by a sufficiently long chain of such scattering events. In this picture, many-body effects are unimportant. The amplification allows one to obtain violations of causality that occur on macroscopic scales. For a sufficiently dilute gas, we detect free particles with no medium effects. The spacetime position of the detected particle can be determined up to experimental uncertainty. With sufficient amplification, the causality violations can be made large on a scale that is large compared to experimental uncertainties.

In this setting, Einstein causality can be reduced to the study of the two-body problem, which can be dealt with in a practical manner.

This type of study can be used to establish the possibility of being able to establish violations of Einstein causality in particle models. In dense matter, there are additional effects that require additional attention. Most notably among are the effects of the Pauli principle and cluster properties. The Pauli principle restricts the allowable final states in a scattering reaction, while cluster properties requires the existence of a number of many-body interactions. The role of these effects is difficult to estimate without a specific model.

III. LONG-WAVELENGTH LIMIT

As noted in the Introduction, the velocity of so-called zero-sound propagation is governed by the pressure-energy relation $(dp/d\epsilon)_s$, where $p(\epsilon)$ is the equation of state. (We assume for this discussion that the system has zero temperature.) The nonrelativistic expression,

$$v^2 = (dp/d\rho)_s, \quad (27)$$

where ρ is the mass density, differs from the relativistic expression by the replacement $\rho \rightarrow \epsilon/c^2$.

The usual procedure is to obtain a relation between the energy density ϵ and the number density n in terms of a dynamical theory. The pressure is then related to the energy density ϵ and the number density n via

$$p = n \frac{d\epsilon}{dn} - \epsilon. \quad (28)$$

A test of causality is to see whether the resulting equation of state $p(\epsilon)$ satisfies $dp/d\epsilon < 1$ (at least for physically reasonable values of ϵ).

Nonrelativistic calculations of the nuclear equation of state have very minimal dependence upon the velocity of light. The energy density can be written as

$$\epsilon = nmc^2 + \Delta\epsilon_{NR}, \quad (29)$$

where m is the mass of the constituent particle and $\Delta\epsilon_{NR}$ includes the kinetic and interaction energy resulting from a calculation which does not depend explicitly upon c . From Eq. 28, it can be seen that the rest energy nmc^2 does not contribute to the pressure, and thus the pressure does not depend upon c . The quantity $\frac{dp}{d\epsilon}$ does depend upon c via the energy density:

$$\frac{dp}{d\epsilon} = \left[nmc^2 + n \frac{d\Delta\epsilon_{NR}}{dn} \right]^{-1} n^2 \frac{d^2\Delta\epsilon_{NR}}{dn^2}. \quad (30)$$

The appearance of c in Eq. 30 is trivial unless $d\epsilon_{NR}/dn$ is comparable to mc^2 , but in that case a nonrelativistic approximation is no longer valid. Thus, the test whether $dp/d\epsilon < 1$ may be less severe than examining the internal consistency of the kinematics leading to the equation of state for nonrelativistic theories of dense matter

IV. BEYOND THE LONG-WAVELENGTH LIMIT

A. Particle theories and wave packet propagation

Local quantum field theories are not necessary for describing an interacting system of particles consistent with Poincaré invariance and quantum mechanics. One can also formulate models using a Hamiltonian, in which particles are the designated degrees of freedom. The dynamics can be specified in terms of a series of direct interactions among $2, 3, \dots$ particles. Cluster properties requires that widely separated clusters of particles behave as they would in complete isolation, the presence of momentum dependent many-body interactions is a consequence.

A strongly interacting relativistic many-body system at high density would therefore satisfy Poincaré invariance and macroscopic causality, but in general it would not satisfy the microscopic causality that is characterized by local interacting fields. Nevertheless, the physical test of such a system is whether it permits signal propagation faster than the velocity of light. Bludman and Ruderman suggest that this can be tested by examining the

poles of the Green function of the interacting system (the sound modes). However, such a Green function is difficult to calculate for a many-body system.

Our purpose in this study is to understand the conditions where a breakdown in causality (in the sense of signal propagation) might occur. To get a preliminary picture, we consider a system of interacting particles in one dimension. One can imagine an incoming projectile striking a constituent particle, which in turn recoils forward, striking a second constituent, which in turn strikes a third, *etc.* Classically, if the particles interact via a rigid barrier (*e.g.*, a string of croquet balls), then causality is violated in the sense that each particle responds instantly to the near approach of a projectile, that is, the time of light propagation across the barrier is not taken into account. For a quantum mechanical system, information propagates via a wave packet.

We will consider two properties of causality with respect to wave packets: the spacetime structure of packet propagation, and the overall time advance of a packet.

In order to study the spacetime structure of packet propagation, it is useful to examine the negative-frequency portion of Pauli-Jordan function in one dimension:

$$\begin{aligned} D_1^-(x) &= \frac{i}{2\pi} \int d^2k e^{-ik \cdot x} \delta(k^2 - m^2) \theta(k^0) \\ &= \frac{1}{2\pi i} \int_{-\infty}^{+\infty} dk \frac{1}{2\omega_m(k)} e^{ikx} e^{-i\omega t}. \end{aligned} \quad (31)$$

After a variable change,

$$k = m \sinh u; \quad \omega = m \cosh u; \quad dk = \omega du, \quad (32)$$

we have

$$D_1^-(x) = \frac{i}{2\pi} \frac{1}{2} \int_{-\infty}^{+\infty} du e^{im(x \sinh u - t \cosh u)}. \quad (33)$$

For the case $x > 0, t > 0, t > x$ (timelike interval), the integral can be reduced as follows:

$$x = \tau \sinh z; t = \tau \cosh z, \quad (34)$$

$$D_1^-(x) = \frac{i}{2\pi} \frac{1}{2} \int_{-\infty}^{+\infty} du e^{-im\tau \cosh u} = \frac{1}{4} H_0^{(2)}(m\tau), \quad (35)$$

where $H_0^{(2)}$ is the zeroeth-order Hankel function of the second kind. For the case $x > 0, t > 0, t < x$ (spacelike interval), the integral can be reduced as follows:

$$x = \lambda \cosh z; t = \lambda \sinh z, \quad (36)$$

$$D_1^-(x) = \frac{i}{2\pi} \frac{1}{2} \int_{-\infty}^{+\infty} du e^{im\lambda \sinh u} = \frac{i}{2\pi} K_0(m\lambda), \quad (37)$$

where K_0 is the zeroeth-order modified Bessel function. For spacelike separation, D^- does not vanish identically, but rather falls off with a range given by the particle Compton wavelength.

In the examples which follow, we will encounter integrals similar to that for D^- , of the form

$$\int_{-\infty}^{+\infty} dk f(k) e^{ikx} e^{-i\omega t}. \quad (38)$$

Even if the integral cannot be evaluated analytically, we can gain some insight by considering the integrand in the complex plane. For spacelike separation ($x > 0, t > 0, x > t$), the phase factor vanishes exponentially as $|k| \rightarrow \infty$ in the upper half plane. If $f(k)$ is analytic in the upper half plane, then the integration contour can be distorted away from the real axis, with the contour “pinned” around the branch cut beginning at $k = +im$. The falloff of the result will therefore be limited by the factor e^{-mx} , *i.e.*, the Compton wavelength scale. This qualitative argument applies to functions $f(k)$ which do not have additional exponential behavior in k . If $f(k)$ has poles or branch points in the upper half plane, then there may be additional scales determining the rate of falloff. In any event, the rate of falloff will be no sharper than the Compton wavelength.

The time evolution of a wave packet is given by

$$\Psi(x', t') = e^{-iHt'} \Psi(x, t = 0). \quad (39)$$

This expression can be evaluated by inserting a complete set of states. In the center of momentum, assuming no bound states, the sum is saturated by scattering states $|k^+\rangle$:

$$\Psi(x', t') = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \langle x' | k^+ \rangle e^{-2i\sqrt{m^2+k^2}t'} \langle k^+ | x \rangle \Psi(x, t=0). \quad (40)$$

The key element in the structure of wave packet propagation is the spectral integral in Eq. 40:

$$\int_{-\infty}^{+\infty} \frac{dk}{2\pi} \langle x' | k^+ \rangle e^{-2i\sqrt{m^2+k^2}t'} \langle k^+ | x \rangle. \quad (41)$$

For non-interacting particles, it has the form of the modified Pauli-Jordan function, as defined in Eq. 38. For interacting systems, we will collect exponential factors in order to use the analysis following Eq. 38.

We can learn about information propagation through a dense system by studying the scattering of two of its constituents. The relevant question for the series of packet backscatterings described above is whether the backscattered packet arrives at a given spatial point “too early” in terms of light propagation time across the range of a direct interaction. This *time advance* of the backscattered packet is in principle an observable effect: each pairwise scattering contributes a time advance ΔT_{adv} , which can accumulate over many scatterings. Whether one can observe a violation of causality on a macroscopic scale depends upon whether information travel time T over a macroscopic distance L is less than L/c when $T = L/v - N\Delta T_{adv}$, where N is the number of pairwise scatterings. Observable consequences would require both v close to c and $Nv\Delta T_{adv}$ close to L (a dense system).

The time advance ΔT_{adv} is determined from the Wigner time delay [10,11] $\Delta T_{delay} = -\Delta T_{adv}$, which can be expressed in terms of the S matrix [12] via

$$\Delta T_{delay} = -i \frac{1}{v} \sum_{\pm} \langle \pm | S(k) | + \rangle^* \frac{d}{dk} \langle \pm | S(k) | + \rangle, \quad (42)$$

where $\langle \pm | S(k) | + \rangle$ is the S matrix for scattering of a particle incident from $x \rightarrow +\infty$ to $x \rightarrow \pm\infty$.

B. Examples

1. Potential Barrier

A simple example is the potential barrier:

$$V(x) = \begin{cases} V_0, & -a \leq x \leq 0 \\ 0, & x < -a, \quad x > 0 \end{cases} \quad (43)$$

The solution is

$$\langle x|k^+\rangle = \psi_k^+(x) = \begin{cases} e^{ikx} + Ae^{-ikx}, & x < -a \\ Be^{ilx} + Ce^{-ilx}, & -a \leq x \leq 0 \\ De^{ikx}, & x > 0 \end{cases} \quad (44)$$

where

$$\begin{aligned} B &= \frac{2(1+\nu)e^{-ika}}{\Delta}; & C &= -B \left(\frac{1-\nu}{1+\nu} \right); \\ A &= \frac{1}{2}e^{-ika}B(1-\nu)(e^{-ila} - e^{ila}); & D &= B + C; \\ l^2 &= k^2 - mV_0; & \nu &= l/k; & \Delta &= (1+\nu)^2e^{-ila} - (1-\nu)^2e^{ila}. \end{aligned} \quad (45)$$

For the case $k^2 < V_0$, $l \rightarrow i\lambda$, where $\lambda^2 = mV_0 - k^2$.

The simplest case to consider is the rigid barrier: $V_0 \rightarrow \infty$. The solution can be written as

$$\psi_k^+(p) = A \sin k(x+a)\theta(x+a). \quad (46)$$

In the spectral sum of Eq. 41, there will appear exponentials associated with free particles,

$$e^{i[\pm k(x-x') - 2\sqrt{m^2+k^2}t']}, \quad (47)$$

as well as exponentials associated with the barrier:

$$e^{i[\pm k(x+x'+2a) - 2\sqrt{m^2+k^2}t']}. \quad (48)$$

The spectral sum over this second set of exponentials will yield a decaying function of range $1/m$ under the condition

$$(x + x' + 2a)^2 > 4t'^2. \quad (49)$$

Thus, there is a range of spacetime events where $(x + x' + 2a)^2 < 4t'^2$, but also $(x + x')^2 > 4t'^2$ (the actual physical causality condition), for which the spectral sum yields oscillating rather than a decaying behavior. This range corresponds precisely to the region of overlap of the barriers. Physically, two hard spheres will repel each other at the instant their edges come in contact. The struck sphere starts moving instantly, leaving no time for information about the interaction to travel across the sphere at the speed of light.

The non-causal behavior illustrated above has a range characterized by the interaction, and goes beyond the Compton wavelength associated with particle masses. How does this non-causal behavior manifest itself in terms of information propagation in a dense system? One way to see this is to examine the propagation of a wave packet which represents a pulse of information. Consider the evolution of a Gaussian packet:

$$\Psi(x, t = 0) = \frac{1}{\sqrt{\sqrt{2\pi}b}} e^{-x^2/2b^2}. \quad (50)$$

The prototype for successive scatterings through a medium is the backscattered portion of a packet in a two-particle subsystem in its center of momentum, as shown in Fig. 1. For a causal system, we expect the centroid of a backscattered packet to arrive at a point x' at a time $t' = (-x' - x)/v$, where $v = k/2\omega_k$ is the velocity in the center-of-momentum frame. Note that both x and x' are negative. The time advance given by Eq. 42 is $\Delta T_{adv} = 2a/v$, *i.e.*, the travel time for the packet to go from $x = -a$ to $x = 0$ and back again, which is eliminated by the presence of the rigid barrier. We have evaluated numerically the packet propagation for various parameters k , a , and b , and find that in all cases considered, backscattered packets reform almost immediately upon collision with the barrier, and reflect the time advance $2a/v$.

For the case of a *finite* barrier, the coefficients A, B, C, D have a complicated dependence on k through l or λ , so it is more difficult to formulate a simple analytic picture. Numerical evaluation of Eq. 42 yields a time advance, though the amount is smaller than $2a/v$.

2. Separable Yamaguchi Potential

The scattering wave function $\psi_k^+(p) = \langle p|k^+ \rangle$ satisfies the equation

$$\psi_k^+(p) = \phi_k(p) + \int_{-\infty}^{+\infty} \frac{dq}{2\pi} G_+(p) V(p, q) \psi_k^+(q), \quad (51)$$

where the Green function reflects the choice of adding interactions to the non-interacting mass squared:

$$G_+(p) = \frac{1}{4(m^2 + k^2) - 4(m^2 + p^2) + i0} = \frac{1}{4(k^2 - p^2) + i0} \quad (52)$$

and

$$V(p, q) = gv(p)v(q) \quad (53)$$

has a separable Yamaguchi form. For definiteness, we use

$$v(p) = \frac{1}{p^2 + \Lambda^2}. \quad (54)$$

Using the Yamaguchi pole form, we get

$$\langle x|k^+ \rangle = e^{ikx} - \frac{gv^2(k)}{D(k)} \frac{i}{8} \left[\frac{1}{k} e^{ik|x|} + \frac{i}{\Lambda} e^{-\Lambda|x|} \right], \quad (55)$$

where

$$D(k) = 1 + \frac{ig}{4} v^2(k) \left[\frac{1}{2k} + \frac{i}{4\Lambda^3} (k^2 + 3\Lambda^2) \right]. \quad (56)$$

Equation 55 is the sum of the incident wave, plus a scattered wave containing two pieces, one obeying “outgoing” boundary conditions (in one dimension), the other having an exponential falloff outside the interaction region.

We now consider the spectral sum in Eq. 41. All of the x dependence appears as explicit exponentials in Eq. 55. The oscillating exponentials, together with the time dependence, yield exponentials of the form

$$e^{i[k(x-x') - 2\sqrt{m^2 + k^2} t']}. \quad (57)$$

While there is additional k dependence multiplying this overall exponential, it has the form given by Eq. 38, and we thus expect the usual causality condition associated with the D and K functions [7] to arise. However, there will also appear cross terms involving oscillating exponentials and decaying exponentials, as well as a bilinear term of the form

$$e^{-\Lambda x} e^{-\Lambda x'}. \quad (58)$$

The integration variable k does not appear in these exponentials, but they do affect the causality condition. For sufficiently small values of t , therefore, these terms will contribute a non-vanishing amount in the acausal region. This acausal effect vanishes for values x and/or x' greater than the scale Λ^{-1} .

While one can see explicitly how this works for a separable Yamaguchi interaction, it seems clear that the real issue is one of scales. Similar effects should occur in more realistic three dimensional quantum mechanical models.

How will a wave packet be affected? While the packet overlaps the interaction region, there will certainly be non-causal contributions. Outside the interaction, however, where values of x or x' less than Λ^{-1} have little support, the non-causal contributions are suppressed. For repulsive potentials ($g > 0$), there is a non-vanishing packet time advance, but its value is smaller than all other scales in the problem.

C. Effect of Pauli Principle

Of course, the examples of two-particle scattering considered here completely ignore the presence of the dense surrounding medium. Apart from an exact treatment of many-body correlation effects, an important effect in a strongly interacting many-fermion system such as dense nuclear matter comes from the fact that scattering into final states below the Fermi surface is forbidden because the states are already filled [13]. Thus, for an incident plane wave in one dimension with momentum below the Fermi surface, forward scattering is permitted but backward scattering is not. In general, backscattered wave packets can only

contain components above the Fermi surface. Such higher momentum components will be less sensitive to the details of the nuclear two-particle potential, and the link to non-causal effects will be commensurately reduced.

D. Packet widths and observable non-causal behavior

While both the local barrier and the Yamaguchi interaction violate a causal condition via the spectral sum in Eq. 41, only the local barrier (finite or infinite) displays a backscattered wave packet advance. In the latter case, the consequences are in principle observable, but one must construct a wave packet with properties suitable for such an observation. The distance scale of the violation is the range a of the interaction (*i.e.*, size of the barrier). In order to observe such a violation for a single scattering, one would need a packet whose overall width is comparable to a . Thus, a nucleon-nucleon core radius of 0.5 fm would correspond to a relevant packet whose momentum spread is roughly $2 \text{ fm}^{-1} = 400 \text{ MeV}/c$, which is quite different from the characteristics of a particle emerging from an accelerator.

To observe non-causal effects in a many-particle system, one could use much wider packets, since the accumulated spacetime advance has a much larger scale. Such a packet would then have a very sharp momentum distribution, and a qualitative picture emerges by considering individual plane-wave states. As noted above, only momentum components above the Fermi surface contribute to backscattering. Furthermore, if the repulsive core is not singular, only a portion of the incident packet will be backscattered. This means that each successive scattering will reduce the amplitude of the packet, making it much more difficult to observe in a macroscopic system.

V. CONCLUSIONS

We have examined a number of issues associated with the nature of causality in theories of dense matter. A strong causality condition is the vanishing of field commutators (or anticommutators for fermions) for spacelike separations of spacetime. This condition

precludes any observable non-causal behavior over arbitrarily small distance scales. In field theories, this condition cannot be achieved without the antiparticle contributions to the field. It should however be noted that the presence of antiparticles does not guarantee locality. The important observation is that the field itself is the appropriate local observable (in the case of Bose fields). The particle excitations of the field are not adequate for testing Einstein causality because there are problems in localizing particle degrees of freedom within a Compton wavelength. This problem, which occurs on the same scale as the violations of local commutation relations when antiparticles are dropped, is related to the inability to define a position operator that transforms as a four-vector. It is a manifestation of the uncertainty principle. In a sense, a relativistic free particle behaves like a composite particle with its size given by its Compton wavelength.

If a dense medium is described via the direct interaction of particles rather than fields, then there are apparent violations of causality on the Compton wavelength scale. This is not a real problem; it is related to the inability to localize the particle within a Compton wavelength. However, without antiparticle degrees of freedom, the theory has no local operators that can be used to precisely test Einstein causality. More seriously, there are violations of causality on the scale of the range interaction that are in principle observable. These come from true non-localities in the dynamics.

Although our discussion and analysis was restricted to models of directly interacting particles, one can anticipate that similar violations of causality can be identified in any truncation of local field theory that leads to a violation of locality. These include the use of cutoffs, effective field approximations, phenomenological quasiparticle models, and Fock space truncations. Clearly, one does not expect that any experiment can lead to violations of Cauchy or Einstein causality; however, our discussion suggests that for any given model, there are in principle some limits to its applicability. Empirically, we find that typical models do not seem to lead to large violations of Einstein causality.

One of the questions which prompted this study is why conventional nonrelativistic calculations of dense matter properties work as well as they do with respect to causal re-

quirements. There appear to be several reasons:

- coincidence: Equations of state $p(\epsilon)$ based upon nonrelativistic dynamics have no *a priori* reason either to satisfy or to violate a causality condition. The dependence upon the speed of light of these calculations appears only trivially through the rest energy contribution to the energy density.
- The densities of interest (say, even up to 10 times normal nuclear density) are still very low compared to the close-packed limit (closer to 100 times nuclear density) where any non-causal effects would be most pronounced.
- Finite range and/or smooth potentials mitigate the time advance characteristic of backscattered wave packets. Repeated backscattering greatly reduces a time-advanced packet of information over macroscopic distances.
- Pauli principle: filled scattering states mean that the backscattered packet is either damped or eliminated.

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FIGURES

FIG. 1. Time advance of a two-particle wave packet in the center-of-mass frame.

